Adjunctions in everyday life

What we talk about when we talk about monads

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Functional Programming

IN

scala

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Foreword by Martin Odersky

MANNING

The Plan

1. I'm going to teach you category theory.
2. I’m going to show you the same pattern several times and say “adjunction” a lot.
3. You’re going to start seeing adjunctions everywhere.
4. You're going to tell me about the adjunctions that you discover.

Categories

$$
\begin{tikzcd}
a \arrow[r, "f"] \arrow[dr, "g\ .\ f"'] & b \arrow[d, "g"] \\
& c
\end{tikzcd}
$$

```
\x -> g (f x)
```

$$
\begin{tikzcd}
a \arrow[r, "f"'] \arrow[rr, bend left, "g\ .\ f"] \arrow[rrr, bend left=60, "h\ .\ g\ .\ f"] & b \arrow[r, "g"'] \arrow[rr, bend right, "h\ .\ g"'] & c \arrow[r, "h"'] & d
\end{tikzcd}
$$

$$
\begin{tikzcd}
a \arrow[r, "id"] \arrow[dr, "f"'] & a \arrow[d, "f"] \\
& b
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
a \arrow[r, "f"] \arrow[dr, "f"'] & b \arrow[d, "id"] \\
& b
\end{tikzcd}
$$

Category

- Objects
- Arrows between objects
- Composition of arrows
    - Which is associative
    - And has an identity

The category *Hask*

- Objects: Haskell types
- Arrows: Haskell functions
- Composition: function composition
  - `λx → f (g (h x))`
  - `λx → x`

Adjunctions

- “Adjoint functors arise everywhere”
- “An adjoint functor is a way of giving the most efficient solution to some problem via a method which is formulaic.”
- Dually, finding the most difficult problem that such a formulaic method solves.

```haskell
curry :: ((a,b) → c) → a → b → c
curry f a b = f (a,b)

uncurry :: (a → b → c) → (a,b) → c
uncurry f (a,b) = f a b
```

$$(a, b) \to c \quad \Leftrightarrow \quad a \to b \to c$$

$$(a, b) \to c \quad \Leftrightarrow \quad a \to b \to c$$

$$f\ a \to c \quad \Leftrightarrow \quad a \to g\ c$$

```haskell
class (Functor f, Functor g) =>
  Adjunction f g where
    leftAdjunct  :: (f a -> b) -> a -> g b
    rightAdjunct :: (a -> g b) -> f a -> b
```

```haskell
class (Functor f, Functor g) =>
  Adjunction f g where
    leftAdjunct  :: (f a -> b) -> a -> g b
    rightAdjunct :: (a -> g b) -> f a -> b
```

The law of adjunctions:

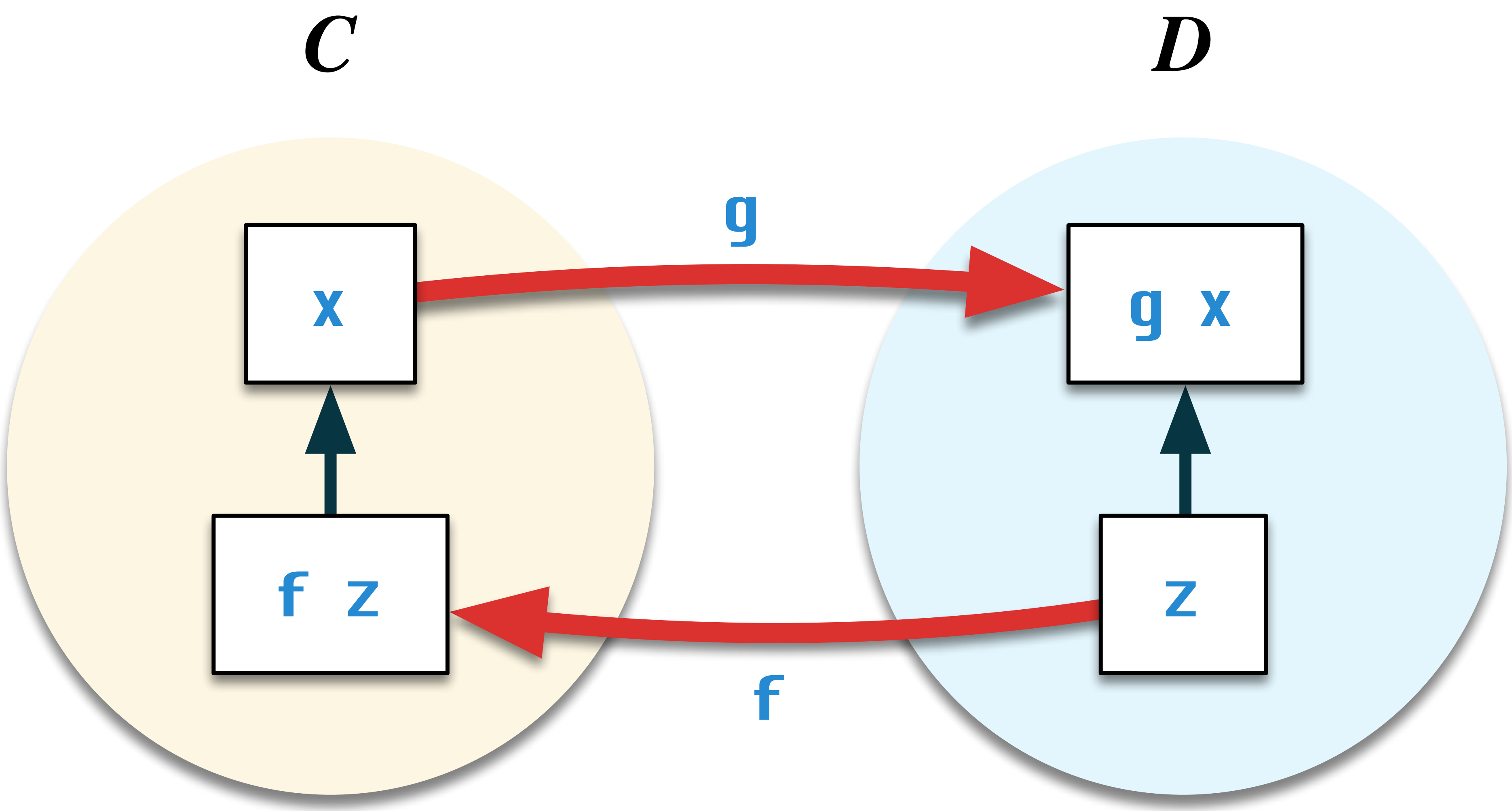
```haskell
leftAdjunct . rightAdjunct = id
rightAdjunct . leftAdjunct = id
```

```haskell
instance Adjunction (,s) (s→) where
  leftAdjunct  = curry
  rightAdjunct = uncurry
```

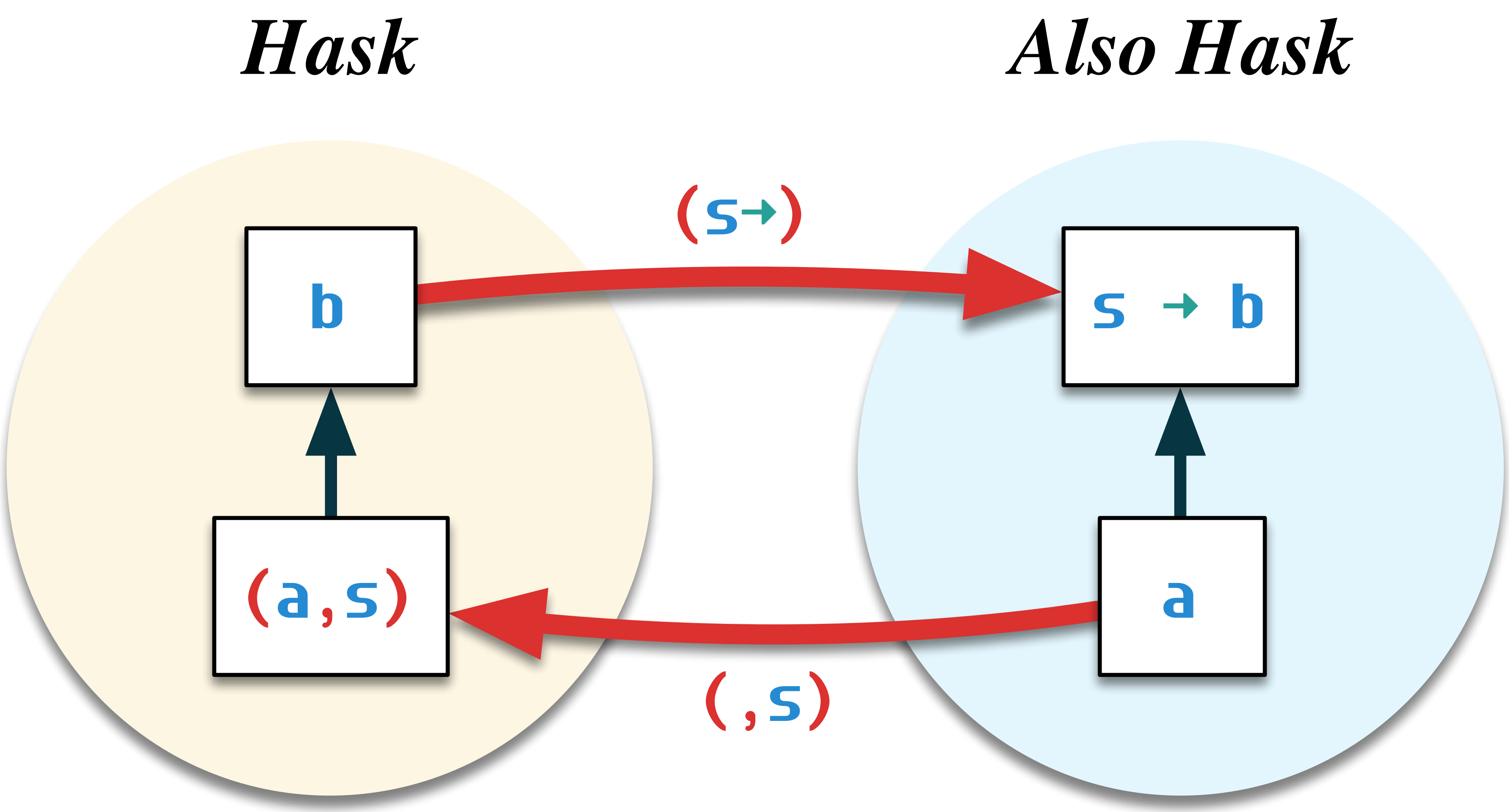
```haskell
class (Functor f, Functor g) =>
  Adjunction f g where
    leftAdjunct  :: (f a -> b) -> a -> g b
    rightAdjunct :: (a -> g b) -> f a -> b

    unit   :: a -> g (f a)
    unit = leftAdjunct id

    counit :: f (g a) -> a
    counit = rightAdjunct id
```

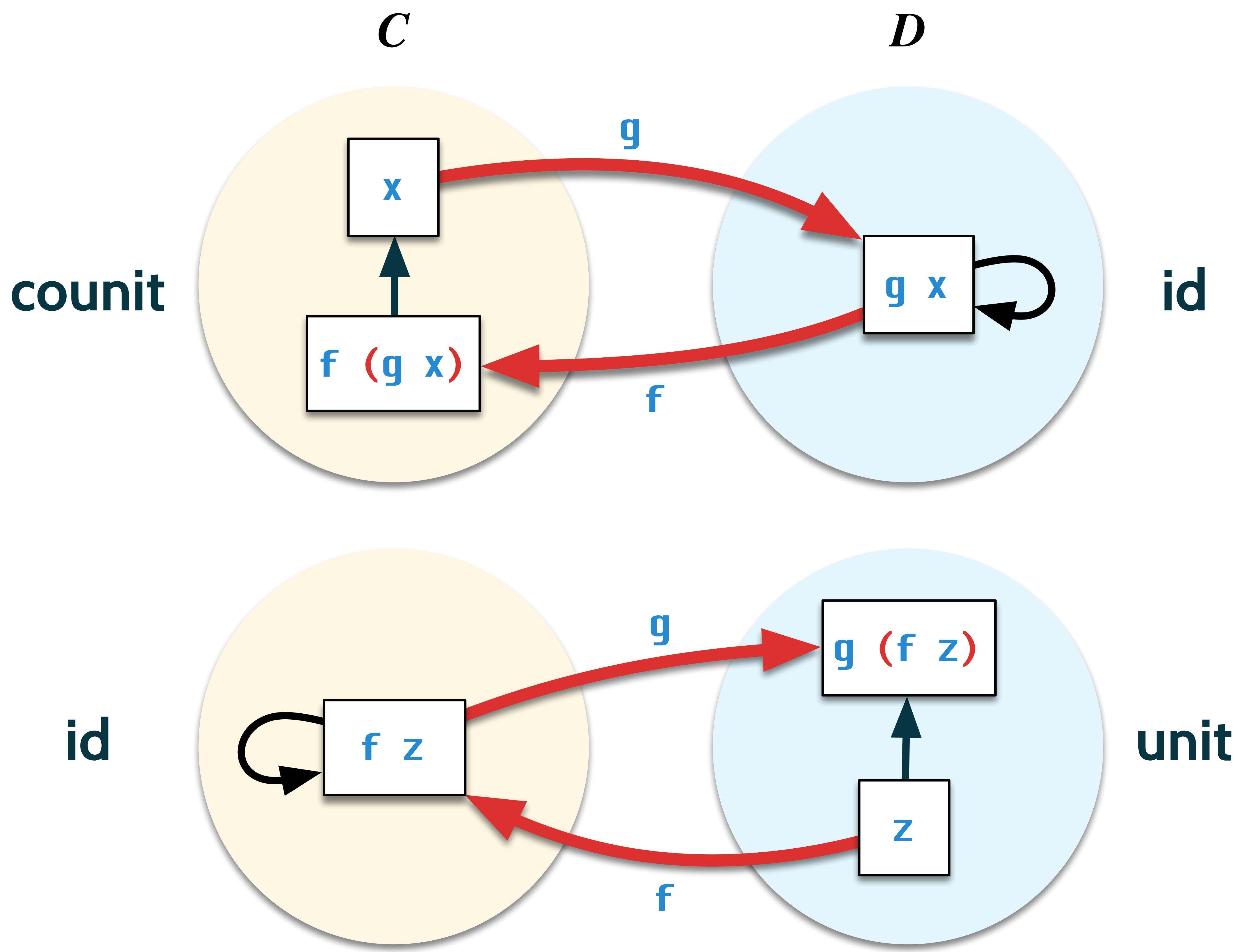
```haskell
class (Functor f, Functor g) =>
  Adjunction f g where
    leftAdjunct  :: (f a -> b) -> a -> g b
    leftAdjunct h = fmap h . unit

    rightAdjunct :: (a -> g b) -> f a -> b
    rightAdjunct h = counit . fmap h

    unit    :: a -> g (f a)
    unit = leftAdjunct id

    counit :: f (g a) -> a
    counit = rightAdjunct id
```

$$f \dashv g$$

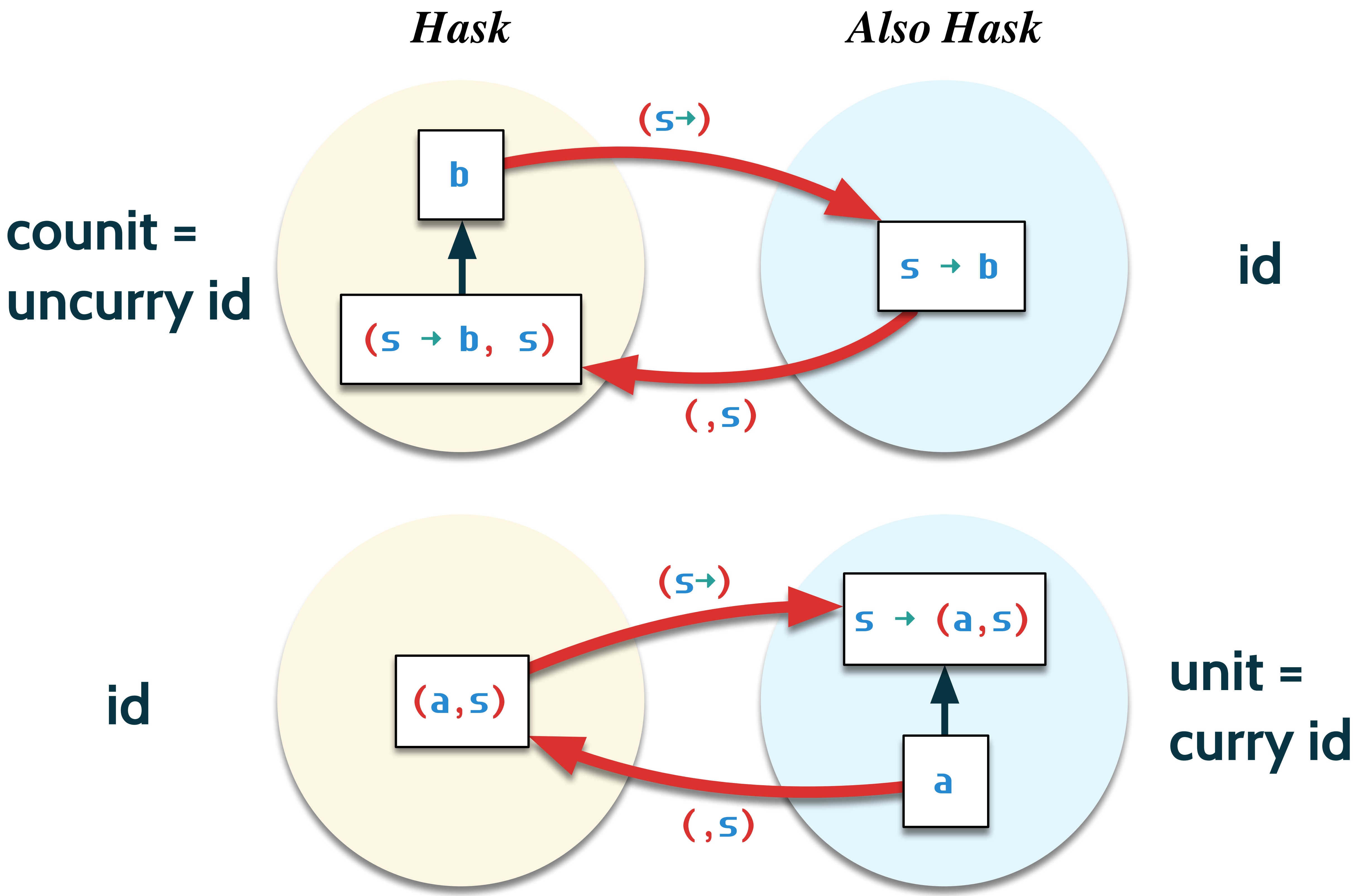


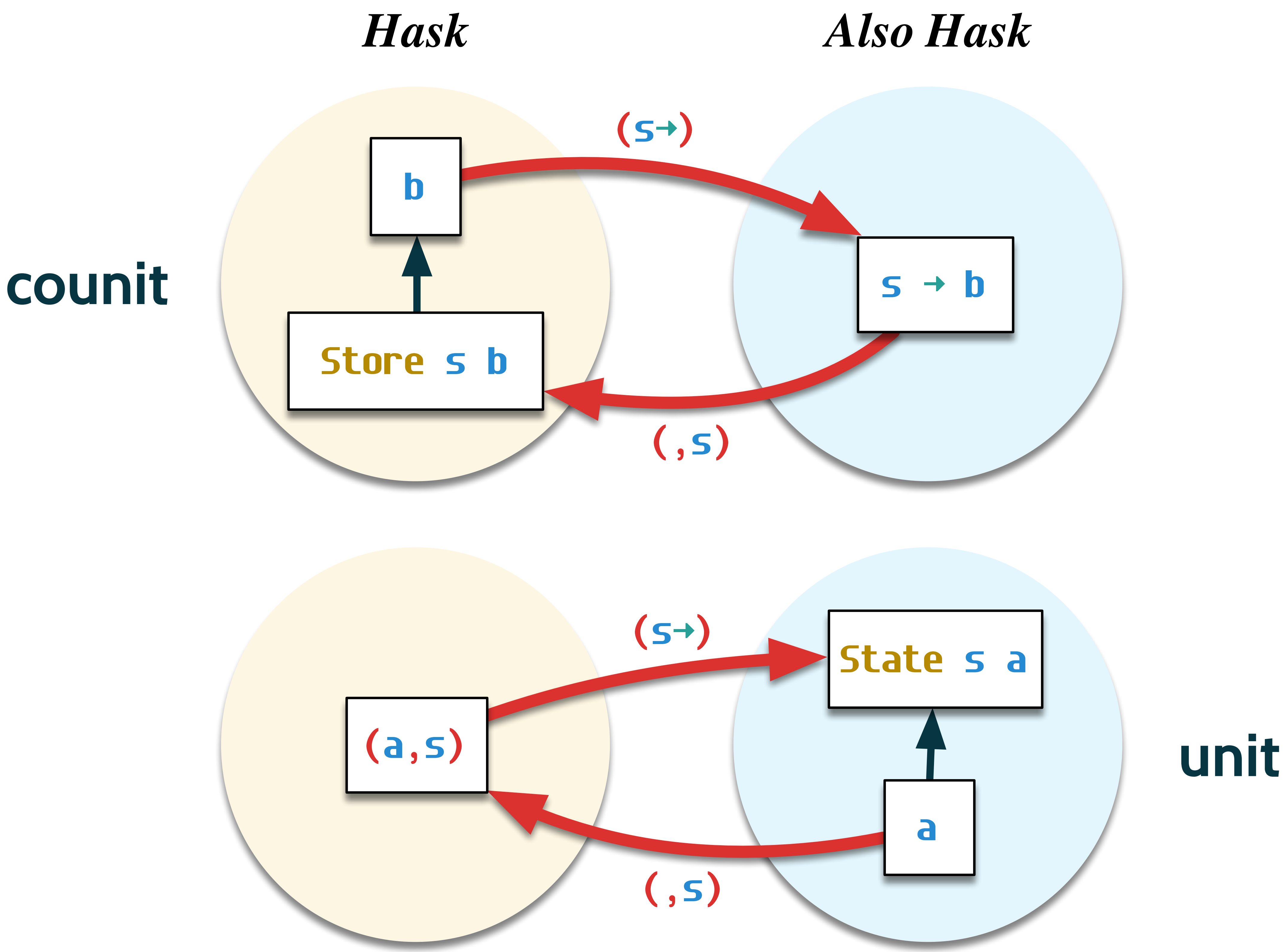
$$(,s) \dashv (s\to)$$





C
D
counit
x
f (g x)
g
g x
f
id
id
f z
g
g (f z)
unit
z
f


Hask
Also Hask
counit =
uncurry id
b
(s→)
s → b
(s → b, s)
(,s)
id
id
(a,s)
(s→)
s → (a,s)
a
(,s)
unit =
curry id



Hask
Also Hask
counit
(s→)
b
s → b
Store s b
(,s)
(s→)
State s a
(a,s)
unit
a
(,s)

state before state after

```haskell
type State s a = s -> (a,s)
```

output

index

cursor

```
type Store s a = (s → a,s)
```

stored value

```
type Bitmap2D = Store (Int,Int) Color
```

$$\texttt{join} :: \overbrace{(s \to (s \to (a,s),s))}^{\substack{\texttt{s → Store s (a,s)} \\ \texttt{State s (State s a)}}} \to \overbrace{s \to (a,s)}^{\texttt{State s a}}$$

```
join = fmap counit
```

```haskell
class (Functor m) => Monad m where
  return :: a -> m a
  join :: m (m a) -> m a

  (>>=) :: m a -> (a -> m b) -> m b
  m >>= f = join $ fmap (unit . f) m
```

Store s (Store s a)

Store s a

(State s (s → a), s)

```
duplicate :: (s → a, s) → (s → (s → a, s), s)
duplicate = fmap unit
```

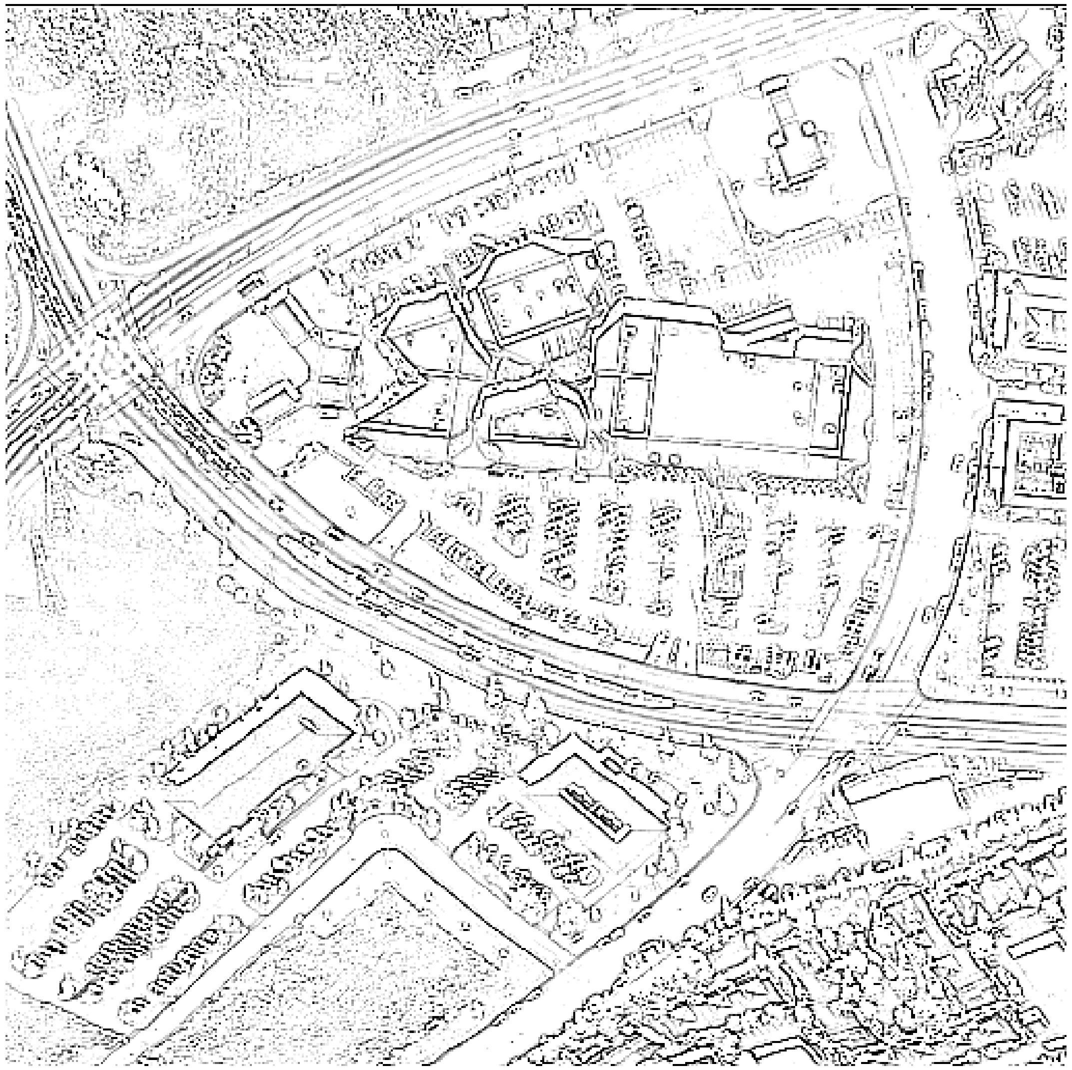
```haskell
class (Functor w) => Comonad w where
  extract   :: w a -> a
  duplicate :: w a -> w (w a)

  (=>>) :: w a -> (w a -> b) -> w b
  w =>> f = fmap f (duplicate w)
```

A comonad extends a local computation to a global one.

```
type Bitmap2D = Store (Int,Int) Int
```

```haskell
lowPass :: Bitmap2D → Bitmap2D
lowPass bmp = bmp =>> mean
```


```haskell
edges :: Bitmap2D → Bitmap2D
edges bmp = bmp =>> \b →
  extract b - extract (lowPass b)
```



$$(a, b) \to c \quad \Leftrightarrow \quad a \to b \to c$$

Category of integers

$$\begin{tikzcd} a \arrow[r, "a \leq b"] \arrow[dr, "a \leq c"'] & b \arrow[d, "b \leq c"] \\ & c \end{tikzcd}$$

$$a \xrightarrow{a \leq a} a$$

$$
\begin{tikzcd}
a \arrow[r, "a \leq b"'] \arrow[rr, bend left, "a \leq c"] \arrow[rrr, bend left=60, "a \leq d"] & b \arrow[r, "b \leq c"'] \arrow[rr, bend right, "b \leq d"'] & c \arrow[r, "c \leq d"'] & d
\end{tikzcd}
$$

```
x,y,z :: Integer
given y > 0

(z*y ≤ x) ⇔ (z ≤ x/y)
```

```
(z*y ≤ x) ⇔ (z ≤ x/y)

unit:    (x/y)*y ≤ x
counit: z ≤ (z*y)/y
```

Galois Connection

$$z * y \leq x \iff z \leq x / y$$

$$f\ z \leq x \iff z \leq g\ x$$

unit: $f\ (g\ x) \leq x$

counit: $z \leq g\ (f\ z)$

Conceptualization as an adjunction

Collections:

$c1 \subseteq c2$ when $c2$ contains all of $c1$

Descriptions:

$d1 \preceq d2$ when $d1$ is more specific than $d2$

$$\mathtt{describe}\ (\mathtt{examples}\ d) \preceq d$$

$$e \subseteq \mathtt{examples}\ (\mathtt{describe}\ e)$$

$$\texttt{describe} \dashv \texttt{examples}$$

$$\texttt{describe}\ e \preceq c \Leftrightarrow e \subseteq \texttt{examples}\ c$$

a simple API design problem

```
indexOf :: Eq a => a -> [a] -> Integer
```

```
(-1)

Infinity

(-Infinity)

NaN

null :: forall a. a
```

```
class Pointed a where
    point :: a
```

Can we turn any type into a pointed type in a formulaic, universal way?

Making no ad hoc choices?

There's a *forgetful* functor
U: ***PointedTypes*** $\to$ ***Types***

U[x] "forgets" the point of x and gives the underlying type.

P $\dashv$ U

U: ***PointedTypes*** → ***Types*** has a *left adjoint:*

P: ***Types*** → ***PointedTypes***

for any type x, P[x] is a pointed type

$$\mathbf{P} \dashv \mathbf{U}$$

$$\mathbf{P}\,a \to b \quad \Leftrightarrow \quad a \to \mathbf{U}\,b$$

$$P\ a \rightarrow b \Leftrightarrow a \rightarrow b$$

```
rightAdjunct :: Pointed b => (a -> b) -> P a -> b

leftAdjunct  :: (P a -> b) -> a -> b
```

```haskell
rightAdjunct :: Pointed b => (a → b) → P a → b
leftAdjunct  :: (P a → b) → a → b

counit :: Pointed b => P b → b
unit   :: a → P a
```

```
rightAdjunct :: b → (a → b) → P a → b
leftAdjunct  :: (P a → b) → a → b

counit :: b → P b → b
unit   :: a → P a
```

```
rightAdjunct :: (a → b) → P a → b → b

leftAdjunct  :: (P a → b) → a → b


counit :: P b → b → b

unit   :: a → P a
```

```haskell
newtype P a = P {
  foldP :: (a → b) → b → b
}

counit = flip foldP id
unit a = P $ λf _ → f a
```

```
foldP :: P a → (a → b) → b → b
```

```
maybe :: b → (a → b) → Maybe a → b
```

```haskell
data Maybe a = Nothing | Just a

maybe :: b → (a → b) → Maybe a → b
maybe b f Nothing = b
maybe b f (Just a) = f a


counit b = maybe b id
unit = Just
```

```haskell
join :: Maybe (Maybe a) → Maybe a
join = counit

duplicate :: Maybe a → Maybe (Maybe a)
duplicate = fmap unit
```

```haskell
indexOf :: Eq a => a -> [a] -> Maybe Integer
```

```
class Monoid a where
    mempty :: a
    mappend :: a → a → a
```

Can we turn any type into a monoid in a formulaic, universal way?

Making no ad hoc choices?

$\mathbf{M} \dashv \mathbf{U}$

U: ***Monoids*** → ***Types*** has a *left adjoint:*

M: ***Types*** → ***Monoids***

for any type x, M[x] is a monoid

```haskell
rightAdjunct :: Monoid b => (a -> b) -> M a -> b

leftAdjunct  :: (M a -> b) -> a -> b

counit :: Monoid b => M b -> b

unit   :: a -> M a
```

```
rightAdjunct :: Monoid b => (a → b) → M a → b
leftAdjunct  :: (M a → b) → a → b
counit :: Monoid b => M b → b
unit    :: a → M a
```

```haskell
class Foldable t where
  foldMap :: Monoid m => (a -> m) -> t a -> m
```

```
foldMap :: Monoid m => (a -> m) -> [a] -> m
```

Can we turn any functor into a monad in a formulaic, universal way?

Making no ad hoc choices?

Free $\dashv$ Forget

Free: ***Monads*** → ***Functors*** has a *left adjoint:*

Forget: ***Functors*** → ***Monads***

for any functor F, Free[F] is a monad

```haskell
rightAdjunct :: Monad m =>
  (forall a. f a -> m a) -> Free f a -> m a

leftAdjunct ::
  (forall a. Free f a -> g a) -> f a -> g a

counit :: Monad m => Free m a -> m a

unit :: a -> Free f a
```

$$\text{Free} \dashv \text{Forget}$$

Free $\dashv$ Forget $\dashv$ Cofree

For two objects **A** and **B** in a category, can we approximate a notion of *"both **A** and **B**"* ?

...that works universally and identically for any **A** and **B**

For any two categories ***C*** and ***D*** there's a product category ***C***x***D*** with

- Objects: Pairs of objects, one from ***C***, one from ***D***
- Arrows: Pairs of arrows, one from ***C***, one from ***D***

Diagonal functor

$\Delta: \boldsymbol{C} \to \boldsymbol{C} \times \boldsymbol{C}$

$\Delta c = [c, c]$

$\Delta f = [f, f]$

$$\Delta \dashv \Pi$$

$$\Delta a \Rightarrow [b,c] \Leftrightarrow a \to \Pi[b,c]$$

$$[a,a] \Rightarrow [b,c] \Leftrightarrow a \to \Pi[b,c]$$

$$[a,a] \Rightarrow [b,c] \Leftrightarrow a \rightarrow b \times c$$

$$(a \to b,\ a \to c) \Leftrightarrow a \to b{\times}c$$

$$(a \to b,\ a \to c) \Leftrightarrow a \to b \times c$$

$$(b \times c \to b,\ b \times c \to c)$$

$$
\begin{tikzcd}
& a \arrow[dl, "\delta_1"'] \arrow[d, dotted] \arrow[dr, "\delta_2"] & \\
b & b \times c \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] & c
\end{tikzcd}
$$

```
fst :: (b,c) → b
snd :: (b,c) → c
```

$$(a \to b,\ a \to c) \Leftrightarrow a \to b \times c$$

$$(a \leq b) \wedge (a \leq c) \Leftrightarrow a \leq b \times c$$

$$(a \leq b) \wedge (a \leq c) \Leftrightarrow a \leq b \times c$$

counit: $(b \times c \leq b) \wedge (b \times c \leq c)$

unit: $a \leq a \times a$

$$(a \geq b) \wedge (a \geq c) \Leftrightarrow a \geq b+c$$

counit: $(b+c \geq b) \wedge (b+c \geq c)$

unit: $a \geq a+a$

$$\mathbf{LUB} \dashv \Delta \dashv \mathbf{GLB}$$

$$\mathtt{Either} \dashv \Delta \dashv (,)$$

$$\Sigma \dashv \Delta \dashv \Pi$$

$$\begin{array}{l} F \dashv G \\ P \dashv Q \\ \hdashline FP \dashv GQ \end{array}$$

Generic Programming with Adjunctions
Ralf Hinze

Galculator: functional prototype of a Galois-connection based proof assistant
Paulo Silva, José Oliveira

What does adjunction *mean*?

- Generates a solution that naturally fits the problem.
- Resolves tension between tradeoffs.
- Finds an optimal surface between a problem space and solution space.

What does adjunction *mean*?

- Compares two categories.
- Simulates one category in another.

Whenever we’re looking for a general, natural, elegant, and efficient solution, we can express the problem as a functor and find its adjoint.

Adjunctions are everywhere.

Let’s find them.

Questions?